PHP2610: Causal Inference & Missing Data

Problem Set 4

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Question 1 (30 points)

Consider a binary treatment variable $A \in \{0, 1\}$, a continuous mediator M, and baseline covariates X. Prove why the following equation holds using the identification assumptions we learned:

$$\mathrm{E}[Y^{1,M^0} - Y^{0,M^0}|X]$$
$$= \int \{\mathrm{E}[Y|A=1, M=m, X] - \mathrm{E}[Y|A=0, M=m, X]\} dF_{M|A=0,X}(m)$$

[Hint 1] The following conditions should be used: for all levels of $a, a', m, Y^{a,m} \perp\!\!\!\perp A|X$; $Y^{a,m} \perp\!\!\!\perp M^{a'}|X, A$; $M^a \perp\!\!\!\perp A|X$, and consistency. Please specify when each of this condition is used in your proof.

[Hint 2] Use the following condition as well: $Y^{a,m} \perp\!\!\!\perp A|M^{a'} = m', X$ for all levels of a, a', m, m'.

Solution

I.

$$\begin{aligned}
\mathrm{E}[Y^{1,M^0}|X] &= \int \mathrm{E}[Y^{1,m}|M^0 = m, X] dF_{M^0|X}(m) \\
&= \int \mathrm{E}[Y^{1,m}|A=0, M^0 = m, X] dF_{M^0|X}(m) && (\text{by } \ Y^{a,m} \perp\!\!\!\perp A|M^{a'} = m', X) \\
&= \int \mathrm{E}[Y^{1,m}|A=0, X] dF_{M^0|X}(m) && (\text{by } \ Y^{a,m} \perp\!\!\!\perp M^{a'}|X, A) \\
&= \int \mathrm{E}[Y^{1,m}|A=0, X] dF_{M^0|A=0,X}(m) && (\text{by } \ M^a \perp\!\!\!\perp A|X) \\
&= \int \mathrm{E}[Y^{1,m}|A=1, X] dF_{M^0|A=0,X}(m) && (\text{by } \ Y^{a,m} \perp\!\!\!\perp A|X) \\
&= \int \mathrm{E}[Y^{1,m}|A=1, M^1 = m, X] dF_{M^0|A=0,X}(m) && (\text{by } \ Y^{a,m} \perp\!\!\!\perp M^{a'}|X, A;\ \text{consistency}) \\
&= \int \mathrm{E}[Y|A=1, M=m, X] dF_{M|A=0,X}(m)
\end{aligned}$$

II.

$$
\begin{aligned}
\mathrm{E}[Y^{0,M^0}|X] &= \int \mathrm{E}[Y^{0,m}|M^0 = m, X] dF_{M^0|X}(m) \\
&= \int \mathrm{E}[Y^{0,m}|A = 1, M^0 = m, X] dF_{M^0|X}(m) \qquad \text{(by } Y^{a,m} \perp\!\!\!\perp A|M^{a'} = m', X) \\
&= \int \mathrm{E}[Y^{0,m}|A = 1, X] dF_{M^0|X}(m) \qquad \text{(by } Y^{a,m} \perp\!\!\!\perp M^{a'}|X, A) \\
&= \int \mathrm{E}[Y^{0,m}|A = 1, X] dF_{M^0|A=0,X}(m) \qquad \text{(by } M^{a} \perp\!\!\!\perp A|X) \\
&= \int \mathrm{E}[Y^{0,m}|A = 0, X] dF_{M^0|A=0,X}(m) \qquad \text{(by } Y^{a,m} \perp\!\!\!\perp A|X) \\
&= \int \mathrm{E}[Y^{0,m}|A = 0, M^1 = m, X] dF_{M^0|A=0,X}(m) \qquad \text{(by } Y^{a,m} \perp\!\!\!\perp M^{a'}|X, A;\ \text{consistency)} \\
&= \int \mathrm{E}[Y|A = 0, M = m, X] dF_{M|A=0,X}(m)
\end{aligned}
$$

III.

$$
\begin{aligned}
\mathrm{E}[Y^{1,M^0} - Y^{0,M^0}|X] &= \mathrm{E}[Y^{1,M^0}|X] - \mathrm{E}[Y^{0,M^0}|X] \\
&= \int \mathrm{E}[Y|A = 1, M = m, X] dF_{M|A=0,X}(m) - \int \mathrm{E}[Y|A = 0, M = m, X] dF_{M|A=0,X}(m) \\
&= \int \{\mathrm{E}[Y|A = 1, M = m, X] - \mathrm{E}[Y|A = 0, M = m, X]\} dF_{M|A=0,X}(m)
\end{aligned}
$$

Question 2 (40 points)

Consider binary treatment A, mediator M, baseline covariates X, and a continuous outcome Y. Suppose the following two regression models are correctly specified.

$$\text{logit}\{P(M = 1|A, X)\} = \alpha_0 + \alpha_1 A + \alpha_2^T X$$
$$\text{E}[Y|A, M, X] = \beta_0 + \beta_1 A + \beta_2 M + \beta_3 AM + \beta_4^T X$$

Assume the sequential ignorability and consistency assumptions hold.

I. (10 points) Derive the conditional controlled direct effect, i.e., $E(Y^{1,0} - Y^{0,0}|X)$.

II. (15 points) Derive the conditional natural direct effect, i.e., $E(Y^{1,M^0} - Y^{1,M^0}|X)$.

III. (15 points) Derive the conditional natural indirect effect, i.e., $E(Y^{1,M^1} - Y^{0,M^0}|X)$.

Solution

I.

$$\begin{aligned}
\text{E}[Y^{1,0} - Y^{0,0}|X] &= \text{E}[Y^{1,0}|A = 1, M = 0, X] - \text{E}[Y^{0,0}|A = 0, M = 0, X] \\
&= \text{E}[Y|A = 1, M = 0, X] - \text{E}[Y|A = 0, M = 0, X] \\
&= (\beta_0 + \beta_1 + \beta_4^T x) - (\beta_0 + \beta_4^T x) \\
&= \beta_1
\end{aligned}$$

II.

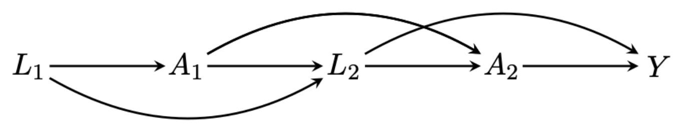
$$\begin{aligned}
\text{E}[Y^{1,M^0} - Y^{0,M^0}|X] = &\sum_m \text{E}[Y|A = 1, M = m, X = x]f(m|X, A = 0) - \\
&\sum_m \text{E}[Y|A = 0, M = m, X = x]f(m|X, A = 0) \\
&= \sum_m f(m|X, A = 0)\{\text{E}[Y|A = 1, M = m, X = x] - \text{E}[Y|A = 0, M = m, X = x]\} \\
&= \sum_m f(m|X, A = 0)\{(\beta_0 + \beta_1 + \beta_2 m + \beta_3 m + \beta_4^T x) - (\beta_0 + \beta_2 m + \beta_4^T x)\} \\
&= \sum_m f(m|X, A = 0)\{\beta_1 + \beta_3 m\} \\
&= \beta_1 \sum_m f(m|X, A = 0) + \beta_3 \sum_m m f(m|X, A = 0) \\
&= \beta_1 + \beta_3 \text{E}[M|X, A = 0] \\
&= \beta_1 + \beta_3 P(M = 1|X, A = 0) \\
&= \beta_1 + \beta_3 \frac{\exp[\alpha_0 + \alpha_2^T X]}{1 + \exp[\alpha_0 + \alpha_2^T X]}
\end{aligned}$$

III.

$$
\begin{aligned}
\mathrm{E}[Y^{1,M^1} - Y^{1,M^0}|X] &= \sum_m \mathrm{E}[Y|A=1, M=m, X=x] f(m|X, A=1) - \sum_m \mathrm{E}[Y|A=1, M=m, X=x] f(m|X, A=0) \\
&= \sum_m \mathrm{E}[Y|A=1, M=m, X=x]\{f(m|X, A=1) - f(m|X, A=0)\} \\
&= \sum_m (\beta_0 + \beta_1 + \beta_2 m + \beta_3 m + \beta_4^T x)\{f(m|X, A=1) - f(m|X, A=0)\} \\
&= (\beta_0 + \beta_1 + \beta_4^T x) \sum_m \{f(m|X, A=1) - f(m|X, A=0)\} + \\
&\quad (\beta_2 + \beta_3) \sum_m m\{f(m|X, A=1) - f(m|X, A=0)\} \\
&= 0 + (\beta_2 + \beta_3)\{\mathrm{E}[M|X, A=1] - \mathrm{E}[M|X, A=0]\} \\
&= (\beta_2 + \beta_3)\left\{ \frac{\exp[\alpha_0 + \alpha_1 + \alpha_2^T X]}{1 + \exp[\alpha_0 + \alpha_1 + \alpha_2^T X]} - \frac{\exp[\alpha_0 + \alpha_2^T X]}{1 + \exp[\alpha_0 + \alpha_2^T X]} \right\}
\end{aligned}
$$

Question 3 (20 points)

Consider the following DAG:



I. `(5 points) Please list all ancestors of $A_2$.`

II. (5 points) Is $Y \perp\!\!\!\perp A_1 | L_1$?

III. (5 points) Is $Y \perp\!\!\!\perp A_1 | A_2, L_2$?

IV. (5 points) Please simplify the decomposition of the joint distribution of (L_1, A_1, L_2, A_2, Y) by removing all unnecessary variables from the conditional part:

$$P(L_1)P(A_1|L_1)P(L_2|L_1, A_1)P(A_2|L_1, A_1, L_2)P(Y|L_1, A_1, L_2, A_2).$$

Solution

I. A_1, L_1, L_2

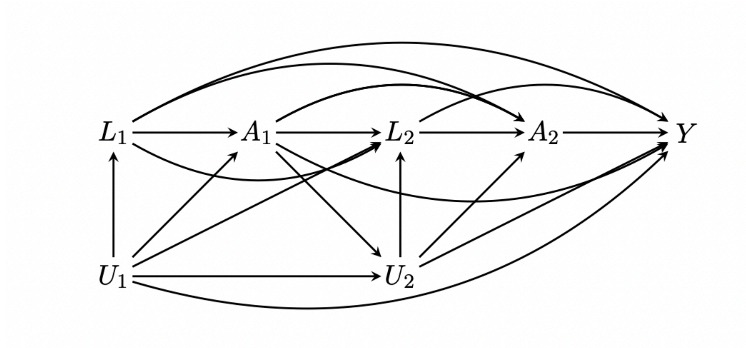
II. No

III. Yes

IV. $P(L_1)P(A_1|L_1)P(L_2|L_1, A_1)P(A_2|A_1, L_2)P(Y|L_2, A_2)$

Question 4 (10 points)

Consider the following DAG:



I. (5 points) Please find at least three non-causal paths between A_1 and Y.

II. (5 points) Does conditioning on L_1 and L_2 satisfy the backdoor criterion to nonparametrically identify the average effect of A_2 on Y? If not, why?

Solution

I. Three non-causal paths between A_1 and Y:

i. $A_1 \to U_2 \leftarrow U_1 \to Y$
ii. $A_1 \to L_2 \leftarrow U_2 \to Y$
iii. $A_1 \leftarrow U_1 \to Y$

II. Conditioning on L_1 and L_2 does not satisfy the backdoor criterion, because it does not close all backdoor paths from A_2 to Y. For example, the path $A_2 \leftarrow A_1 \to Y$ is a valid backdoor path in that it does not contain a node in the conditioning set and it is not blocked by a collider.